

# Quantification of Symmetry

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Symmetry is conventionally described in a contrariety manner that the system is either completely symmetric or completely asymmetric. Using group theoretical approach to overcome this dichotomous problem, we introduce the degree of symmetry (DoS) as a non-negative continuous number ranging from zero to unity. DoS is defined through an average of the fidelity deviations of Hamiltonian or quantum state over its transformation group  $G$ , and thus is computable by making use of the completeness relations of the irreducible representations of  $G$ . The monotonicity of DoS can effectively probe the extended group for accidental degeneracy while its multi-valued natures characterize some (spontaneous) symmetry breaking.

PACS numbers: 03.65.Fd, 11.30.Qc, 02.20.Bb, 03.65.-w

*Introduction.*— Symmetry is a theme of modern physics, which plays a crucial role in the understanding of fundamental interactions of the microscopic world [1] as well as the emergence of macroscopic orders [2]. It has become evident that both the elementary particle structure and the emergent phenomena, e.g., superconductivity and Bose-Einstein condensation, are originated from symmetry and its spontaneous breaking [3–5]. Its applications range from particle physics [6–9] to condensed matter physics [10, 11], and even to biological systems [12, 13].

Conventionally, symmetry is dealt in a dichotomous fashion that a physical system either possesses or not possesses a symmetry. In the group theoretical approach, the symmetry of a quantum system is usually considered by checking that if the system is invariant or not under some transformations, which sometimes form a symmetry group  $G$ . The symmetry breaking of the system can be described as a reduction of the symmetry group to its subgroup. Although this conventional approach has succeed in classifying the spectrum structure and even various phases of matters, it is not natural for us because there is not a room for the intermediate circumstance, namely, a continuous measure of symmetry has not been found. Actually, such intermediate issues exist objectively and needs to be properly quantified. For example, a charged particle moving in a central potential possesses  $SO(3)$  symmetry. When a static magnetic field is applied, no matter how weak it is, the  $SO(3)$  symmetry is said to be broken into  $SO(2)$ . However,  $SO(3)$  symmetry can still be approximately used to simplify the equations describing the dynamics and the energy level structure when the magnetic field is weak enough. Another example is the nuclear system that possesses the isospin  $SU(2)$  symmetry and thus its energy spectrum of

strong interaction can approximately, but effectively, be classified, although the electromagnetic force could break this  $SU(2)$  symmetry.

In this regard, it would be of much interest to present a quantitative description of symmetry and its (spontaneous) breaking in this intermediate circumstance, which could determine the extent of approximation for using a given symmetry in practice. To this end, we, in this letter, introduce a continuous measure of symmetry, i.e., the degree of symmetry (DoS), by considering that symmetry is a relative concept: the particular subset  $G$  of all physically-allowed transformations needs to be specified for assigning a symmetry to a physical system. More specifically, for a given set  $G$  of transformations on the Hamiltonian or the quantum state  $F = H$ , or  $\rho$ , we first define a dual of DoS, the degree  $A(G, F)$  of asymmetry (DoAS), by averaging the fidelity deviations (see definition below) over  $G$ . Generally, the DoAS ranges from zero to unity, and thus the DoS  $S(G, F) = 1 - A(G, F)$  also satisfies  $0 \leq S(G, F) \leq 1$ . Evidently,  $S(G, F)$  offers symmetry an intermediate description to avoid the dichotomy in the conventional group theoretical analysis.

We will show that, if we chose  $G$  as a group, the DoS, bounded with  $1/2 \leq S(G, F) \leq 1$ , facilitates a general computable measure of symmetry based on the irreducible representations of  $G$ . It is potential in identifying various natures of symmetry that are important to emergent phenomena, such as the spontaneous symmetry breaking (SSB). For example, the thermodynamic SSB corresponds to multi-valued natures of DoS at the low temperature, which is similar to the depiction of the spontaneous magnetization [14]. It is also shown that the multi-level crossing by a proliferation of energy levels brings a peak to the DoS and the extended group can be given to account for the hidden symmetry from

accidental degeneracy.

*Degree of symmetry.*— We consider a quantum system with Hamiltonian  $H$ , and a set  $G$  of  $n_G$  transformations on its Hilbert space  $\mathcal{H}$ . When  $OHO^{-1} = H$  for  $O \in G$ , we say that  $H$  (the quantum system) is symmetric with respect to the transformation  $O$ . Actually, all symmetric transformations form a group  $G'(\subset G)$ . It is obvious that, the deviations of  $OHO^{-1}$  from  $H$  measure the extent of the asymmetry of  $H$  with respect to the transformation set. Thus, we use their average over  $G$  to define the degree of symmetry breaking (asymmetry) DoAS

$$A(G, H) = \frac{1}{4|\tilde{H}|^2} \overline{|[R(g), H]|^2} \quad (1)$$

where  $|O| = \sqrt{\text{Tr}\{O^\dagger O\}}$  indicates the Frobenius norm [15] while  $\overline{f(g)}|_G \equiv \overline{f(g)} = n_G^{-1} \sum_{g \in G} f(g)$  is an average of a (group) function  $f(g)$  defined on  $G$ , and later the subscript  $G$  will be occasionally omitted; If  $G$  is a group, then  $R: g \rightarrow R(g) \in \text{End}(\mathcal{H})$  is a  $d$ -dimensional representation of  $g \in G$ . Otherwise,  $R(g)$  represents a unitary transformation on  $\mathcal{H}$ . Here,  $\overline{|[R(g), H]|^2} = \overline{|R(g)^\dagger H R(g) - H|^2}$  is the fidelity deviation of  $H$  under the action of  $g$ , and  $\tilde{H} = H - d^{-1} \text{Tr}\{H\}$  is a re-biased Hamiltonian such that it is invariant under the zero-point energy shifting  $H \rightarrow H + \epsilon$  for  $\epsilon$  being a real number.

It is easy to prove that  $0 \leq A(G, H) \leq 1$ , thus the DoS defined by  $S(G, H) = 1 - A(G, H)$  or

$$S(G, H) = \frac{1}{4|\tilde{H}|^2} \overline{|[R(g), \tilde{H}]|^2} \quad (2)$$

ranges from zero to unity and thus quantifies the extent of the symmetry of  $H$  with respect to  $G$ . The above definition of DoS is evidently reasonable in physics since it possesses the following properties (for the proofs see the supplemental material [16]): (1) Tighter bound when  $G$  forms a transformation group  $0 \leq A(G, H) \leq 1/2 \leq S(G, H) \leq 1$ ; (2) Independence of DoS on the basis, i.e.,  $S(WGW^\dagger, WHW^\dagger) = S(G, H)$ , where  $W$  is a unitary transformation and  $WGW^\dagger = \{WR(g)W^\dagger | g \in G\}$ ; (3) Scaling invariance, i.e.,  $S(G, \lambda H) = S(G, H)$ ; (4) Independence of the choice of the zero-point energy, i.e.,  $S(G, H + \epsilon) = S(G, H)$ ; (5) Hierarchy property  $n_{G'} S(G_s, H) \leq n_G S(G, H)$  for a subset  $G_s(\subset G)$  with  $n_{G'}$  elements.

When  $G$  becomes a group, in the spaces  $\mathcal{H}^{(l)}$  of its  $l$ th irreducible representations with finite dimensions  $d_l$ , the DoS  $S = S(G, H)$  is re-expressed as

$$S = \frac{1}{2} + \sum_l \frac{1}{2d_l} \left( \sum_\alpha \frac{\langle l, \alpha | H | l, \alpha \rangle}{|\tilde{H}|} - \frac{\text{Tr}\{H\}d_l}{d|\tilde{H}|} \right)^2, \quad (3)$$

where  $|l, \alpha\rangle$  is a basis vector of  $\mathcal{H}^{(l)}$  ( $\alpha = 1, 2, \dots, d_l$ ), and we have used the completeness relations of irreducible representations [16]. The bisection point  $1/2$  from property (1) is also reflected in above equation, since each

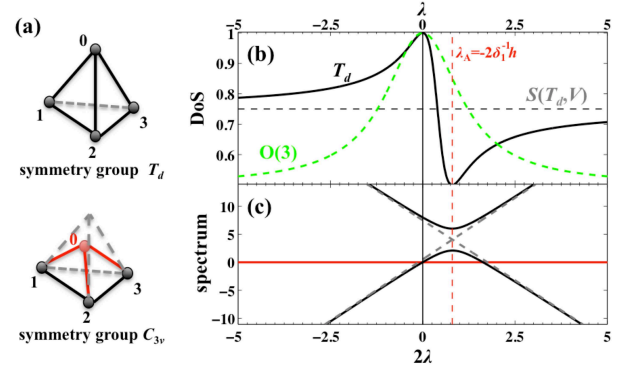


Figure 1: (color online). (a) Top: Schematic of a four-site lattice arranged into the regular tetrahedron geometry, with Hamiltonian  $H$  and symmetry group  $T_d$ . Bottom: The  $T_d$  symmetry is broken into  $C_{3v}$  upon adding the perturbation  $H' = \lambda V$ , which changes the hopping strength as well as the site energy relevant for the 0th site. (b) Degree of symmetry (DoS) vs  $\lambda$  for the four-site model (black solid) and the angular momentum model (green dashed). The asymptotic value  $S(T_d, V)$  (black dashed) and the local minimum  $\lambda_A$  (red vertical line) for the four-site model are also shown. (c) Energy spectrum of the four-site model vs  $2\lambda$ . Red line indicates the two degenerate  $E$  levels. Avoid level crossing of the two  $A_1$  levels is shown by the grey dashed lines.

term contributes non-negatively in the summation over  $l$ . We point out that, by using Eq.(3), DoS is feasible to be computed based on the measurements with respect to the basis  $\{|l, \alpha\rangle\}$ . Otherwise, for a continuous group, a straightforward calculation of DoS from Eq.(2) should need to carry out the group integral with the Haar measure, e.g., the sum over  $\text{SO}(3)$  becomes a Lie group integral [16].

*Symmetry breaking.*— Let  $G$  be a symmetry group of the quantum system with Hamiltonian  $H$ . A perturbation  $H' = \lambda V$  breaks the symmetry into the subgroup  $G_s(\subset G)$ , i.e.,  $[V, R(g)] \neq 0$  for  $g \in G - G_s$  and  $[V, R(g')] = 0$  for  $g' \in G_s$ . For the total Hamiltonian  $H(\lambda) = H + \lambda V$ , we calculate the DoS under the symmetry breaking [16]

$$S(G, H(\lambda)) = 1 - \frac{A(G, V)\lambda^2}{\lambda^2 + \xi\lambda + \eta}, \quad (4)$$

where  $A(G, V)$  is the DoAS of the Hermitian operator  $V$ , the other two coefficients are defined as  $\xi = 2\text{Tr}\{\tilde{H}\tilde{V}\}|\tilde{V}|^{-2}$  and  $\eta = |\tilde{H}|^2|\tilde{V}|^{-2}$ . The above equation exactly reflects the duality between symmetry and asymmetry: the maximal symmetry breaking due to the perturbation corresponds to the minimal symmetry of the considered system. When  $|\lambda|$  is increased, there exists a special point  $\lambda_A = -2\xi^{-1}\eta$  where the DoS reaches a local minimum  $S_{\min} = 1 - A(G, V) \csc^2 \varphi$ ; here  $\varphi$  is the angle between  $\tilde{H}$  and  $\tilde{V}$  [16].

The following two examples are used to illustrate the above conception on quantifying the extent of symmetry

and its breaking. First, let us consider a particle residing on a four-site lattice with the following Hamiltonian

$$H = \sum_i \epsilon |i\rangle\langle i| + \sum_{ij} h |i\rangle\langle j|, \quad (5)$$

where  $|i\rangle$  ( $i = 0, 1, 2, 3$ ) is the single particle state with site  $i$  occupied. The site energy  $\epsilon$  and the hopping strength  $h$  are site-independent for the regular tetrahedron geometry [see Fig. 1a], and thus  $H$  is symmetric to all transformations from the  $T_d$  group, which contains (combined) rotations and mirror reflections sending a regular tetrahedron into itself [17]. In this example, we let the symmetry  $T_d$  break into  $C_{3v}$  through the following perturbation

$$H' = \lambda[\delta_0|0\rangle\langle 0| + \delta_1\sum_{i=1}^3(|i\rangle\langle 0| + h.c.)], \quad (6)$$

where  $\lambda\delta_0$  and  $\lambda\delta_1$  are the deviations of the energy and the coupling related to the 0th site. It is well known that  $C_{3v}$  has two one-dimensional irreducible representations  $A_1$  and  $A_2$ , as well as one two-dimensional irreducible representation  $E$  [17], which correspond to the three kinds of energy levels with one or two-fold degeneracies.

The above symmetry breaking from  $T_d$  to  $C_{3v}$  is quantified by the DoS through Eq.(4) with  $G = T_d$ . Straightforward calculation shows exact results  $A(T_d, V) = (2\gamma^2 + 16)^{-1}(\gamma^2 + 4)$ ,  $\xi = 16(\gamma^2 + 8)^{-1}\delta_1^{-1}h$ , and  $\eta = \xi\delta_1^{-1}h$ . Here,  $\gamma = \delta_1^{-1}\delta_0$  is the ratio between the two parameters in  $H'$ . As shown in Fig. 1b, the DoS reaches unity when  $\lambda = 0$ , indicating the full  $T_d$  symmetry that possessed by the original Hamiltonian  $H$ . The symmetry breaking perturbation  $H'$  suppresses the DoS first quadratically in  $\lambda$  and then, as  $|\lambda|$  further increased to approach the strong perturbing limit ( $|\lambda| \rightarrow \infty$ ), reaches a  $\gamma$ -dependent asymptotically value  $(2\gamma^2 + 16)^{-1}(\gamma^2 + 12)$ .

In this model, the special point  $\lambda_A = -2\delta_1^{-1}h$ , where the DoS reaches the local minimum, indicates an avoid level crossing in the energy spectrum. To see this, we rewrite  $H(\lambda)$  in terms of the standard basis of irreducible representations by using the projection operator method [17, 18]. The resulting four-dimensional Hilbert space contains two  $A_1$ -representations and one  $E$ -representation of  $C_{3v}$  [17]. The two levels that transform according to the two  $A_1$ -representations are coupled and the corresponding avoid level crossing point  $\lambda_*$  is related to  $\lambda_A$  by

$$\lambda_* = \frac{6 - \gamma}{12 + \gamma^2} \lambda_A. \quad (7)$$

Especially, for  $\delta_0 \ll \delta_1$  the avoid level crossing happens approximately at  $\lambda_A/2$  [see Fig. 1b,c].

Another example demonstrates the DoS of the breaking of the continuous symmetry. The system we considered is a particle with angular momentum  $j$ , whose Hamiltonian reads

$$H = \epsilon J^2, \quad H' = \lambda J_z, \quad (8)$$

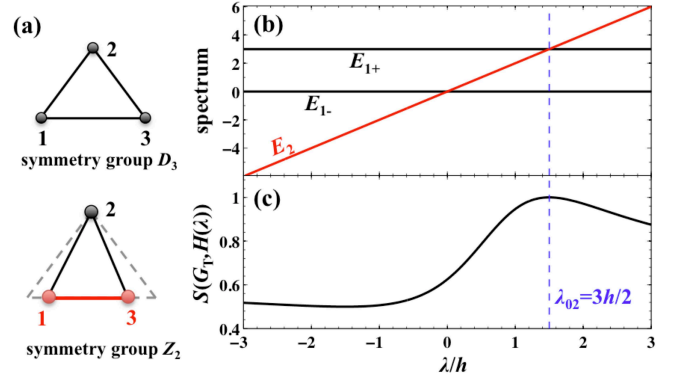


Figure 2: (color online). (a) Schematics of the three-site model, with symmetry  $D_3$  breaking into  $Z_2$  by the perturbation Eq.(10). (b) Energy spectrum vs  $\lambda/h$  for the three-site model, showing two accidental degeneracies at  $\lambda_{01}$  and  $\lambda_{02}$  (blue vertical line). (c) DoS vs  $\lambda/h$  with respect to  $G_T$ , showing that the accidental degeneracy at  $\lambda_{02}$  is identified with the maximum of the DoS.

where  $J_i$  ( $i = x, y, z$ ) are components of the angular momentum operator and  $J^2 = J_x^2 + J_y^2 + J_z^2$ . In this model, the  $O(3)$  symmetry of  $H$  is broken by the perturbation, described by  $H'$ , to  $O(2)$ . With  $G = O(3)$ , the DoS is calculated as  $1 - [2\lambda^2 + \epsilon^2 j(j+2)]^{-1}\lambda^2$  [16]. Unlike the previous model, here the DoS does not show a local minimum and decays monotonically as  $|\lambda|$  increasing. Comparison with the generic result Eq.(4) indicates the underlying condition  $\text{Tr}\{\tilde{H}\tilde{V}\} = 0$ , which is fulfilled by the Hamiltonian Eq.(8).

*Accidental degeneracy.*— Accidental degeneracy of energy levels appears in a quantum system when its parameters are changed to cause a level crossing. It is usually not relevant to the geometric symmetry, but our DoS can reveal the existence of the hidden symmetry. Actually, accidental degeneracy also implies symmetry. The greater the degeneracy, the greater the symmetry.

For the general Hamiltonian  $H(\lambda)$  defined above, we introduce the additional transformations: the  $U(2)$  operations on the two  $\lambda$ -dependent energy levels of  $H(\lambda)$  (or  $U(N)$  operations for the more general  $N$  levels crossing), which will become degenerate as  $\lambda$  tuned to  $\lambda_0$ . Because  $H(\lambda_0)$  is proportional to the identity operator in the degenerate subspace and, as a result, commuted with all  $U(2)$  operations, the symmetry group  $G$  of  $H(\lambda_0)$  is extended to a larger one  $G_T = \langle G, U(2) \rangle$ , which is generated by elements in  $G$  and  $U(2)$ . It is expected that the behavior of DoS could manifest the hidden symmetry that implied by the enlarged group  $G_T$ : the level crossing at  $\lambda_0$  could result in a local dip in the DoAS, when the parameter  $\lambda$  is tuned close to  $\lambda_0$ . To see this, we expand the Hamiltonian linearly around  $\lambda_0$ , i.e.,  $H(\lambda) \approx H(\lambda_0) + \partial_\lambda H(\lambda_0)(\lambda - \lambda_0)$ . Since  $[R(g), H(\lambda_0)] = 0$  for  $g \in G_T$ , the DoAS is written as

$$A(G_T, H(\lambda)) \propto A(G_T, \partial_\lambda H(\lambda_0))(\lambda - \lambda_0)^2. \quad (9)$$

Thus, by the duality, the accidental degeneracy indeed manifests itself as a local maximum at  $\lambda_0$  in DoS.

To illustrate the above idea, we consider the following three-site model whose Hamiltonian is of the same form as Eq.(5) except that  $i \in \{1, 2, 3\}$ . And the perturbation term

$$H' = \lambda[|1\rangle\langle 1| + |3\rangle\langle 3| - (|1\rangle\langle 3| + h.c.)] \quad (10)$$

breaks the symmetry from  $D_3$  to  $Z_2 = \{e, \sigma\}$ . Here, the transformation  $\sigma$  interchanges the basis state  $|1\rangle$  with  $|3\rangle$ . The energy spectrum of  $H(\lambda)$  contains two  $\Gamma_1$  levels  $E_{1\pm} = \epsilon + h/2 \pm \lambda_{02}$  and one  $\Gamma_2$  level  $E_2 = \epsilon - h + 2\lambda$ , where  $\Gamma_{i=1,2}$  are two irreducible representations of  $Z_2$ . The spectrum shows two accidental degeneracies between the  $\Gamma_1$  and the  $\Gamma_2$  levels at  $\lambda_{01} = 0$  and  $\lambda_{02} = 3h/2$ , respectively [see Fig. 2b].

Indeed, at the accidental degeneracy,  $H(\lambda_{02})$  becomes more symmetric since there exists the additional symmetric transformations of  $U(2)$ :  $R(\omega_0; \hat{n}, \omega) = \exp[i(\omega_0 - \hat{n} \cdot \vec{s}\omega)]$  with pseudo spin-1/2 operators  $\vec{s}$  defined by  $s_x = (|\psi_{1+}\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_{1+}|)/2$  et al.. Here,  $|\psi_m\rangle$  is the eigenstate associated with level  $E_m$ . The extended symmetry group  $G_T$  for  $H(\lambda_{02})$  is still  $U(2)$  since  $Z_2 \subset U(2)$  [16]. Thus, the two-fold degenerate subspace supports a two-dimensional irreducible representation of  $G_T$ . It is shown that the DoS  $S(G_T, H(\lambda)) = 1 - 3[\lambda^2 - \lambda_{02}\lambda + \lambda_{02}^2]^{-1}(\lambda - \lambda_{02})^2/8$  reaches the unity when  $\lambda = \lambda_{02}$  [see Fig. 2c]. Therefore, the DoS indeed signals the hidden symmetry. We notice that, without the geometric symmetry, the above  $U(2)$  symmetry defined in the subspace spanned by  $|\psi_{1+}\rangle$  and  $|\psi_2\rangle$  at the accidental degenerate point is similar to the dynamical symmetry  $SO(4)$  of the non-relativistic hydrogen atom [19].

*DoS of quantum state and spontaneous symmetry breaking.*— In emergent phenomena, the symmetry of the system ground state can be different from that of the underlying Hamiltonian or Lagrangian. This difference is roughly regarded as the spontaneous symmetry breaking (SSB) [2, 5, 9]. For a better depiction of those phenomena, we need to introduce the DoS of quantum state (DoSS)  $\rho$ , which is analog to the DoS of Hamiltonian by the Eq.(2)

$$S(G, \rho) = \frac{1}{4|\rho|^2} \overline{|\{R(g), \rho\}|^2} \quad (11)$$

where  $\rho$  is the density matrix of a quantum state. It possesses the similar properties (1)-(5) except for  $S(G, \rho) = S(G, \rho + \epsilon)$ , which we need not to require for physics.

We now use DoSS to characterize the SSB in thermodynamics. We consider the thermalization of a quantum system with degenerate ground states  $\{|G_\alpha\rangle | \alpha = 1, 2, \dots, d_G\}$ , i.e.,  $H|G_\alpha\rangle = \epsilon_0|G_\alpha\rangle$  [20]. At the zero temperature such system will have a non-vanishing entropy  $S = k_B \ln d_G$ , known as the modified third law of thermodynamics [14]. By introducing a perturbation  $H' = \lambda V$

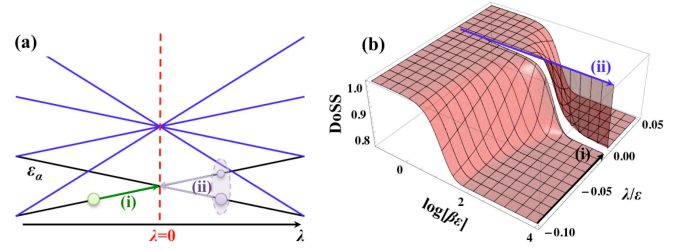


Figure 3: (color online). (a) Energy spectrum of a system with degenerate ground states  $\{|G_\alpha\rangle\}$  at  $\lambda = 0$ . Upon whether  $T \rightarrow 0$  (i) before or (ii) after  $\lambda \rightarrow 0$ , the thermal equilibrium state Eq.(12) approaches different final states. (b) DoS of quantum state (DoSS)  $S(O(3), \rho)$  vs  $\lambda/\epsilon$  and  $\beta\epsilon$  for the angular momentum model. The multi-valued natures of the DoSS at  $T = 0$  and  $\lambda = 0$  are reflected as the two non-commuting limiting processes indicated by (i) and (ii).

to break the symmetry so that  $|G_{\alpha=0} = G_0\rangle$  becomes the unique ground state, the thermodynamic SSB is described as the following two non-commutative limiting processes: (i)  $T \rightarrow 0$  and then  $\lambda \rightarrow 0$ ; (ii)  $\lambda \rightarrow 0$  and then  $T \rightarrow 0$ . In these two non-commutative limiting processes, the following state

$$\rho = \frac{1}{Z} \sum_{\alpha \neq 0} e^{-\epsilon_\alpha/T} |G_\alpha\rangle\langle G_\alpha| + \frac{1}{Z} e^{-\epsilon_0/T} |G_0\rangle\langle G_0| + \dots \quad (12)$$

will approach to  $\rho_{f1} = |G_0\rangle\langle G_0|$  and  $\rho_{f2} = d_G^{-1} \sum_{\alpha} |G_\alpha\rangle\langle G_\alpha|$  respectively [see Fig. 3a].

To see the quantitative details of such thermodynamic SSB, we use the DoSS defined by Eq.(11). Let  $G$  be a symmetry group of  $H$ , and  $\{|G_\alpha\rangle\}$  span an irreducible representation of  $G$ . Because  $\rho_{f2}$  is proportional to the identity operator, thus  $[R(g), \rho_{f2}] = 0$ . This implies that the limiting process (ii) results in a final state with  $S(G, \rho_{f2}) = 1$ . On the other hand, from Schur's theorem [17], in the limiting process (i) there always exists some  $g \in G$  such that  $[R(g), \rho_{f1}] \neq 0$  and consequently  $S(G, \rho_{f1}) < 1$ . This in turn implies an SSB since the final state  $\rho_{f1}$  does not retain the full symmetry of the underlying microscopic Hamiltonian  $H$ .

The above relation between the DoSS and  $\rho_{f1,2}$  suggests the multi-valued natures of DoSS at  $(T = 0, \lambda = 0)$  upon different limiting processes as an SSB witness

$$\lim_{\beta \rightarrow \infty} \lim_{\lambda \rightarrow 0} S(G, \rho) = 1, \quad \lim_{\lambda \rightarrow 0} \lim_{\beta \rightarrow \infty} S(G, \rho) < 1, \quad (13)$$

where  $\beta = 1/T$  is the inverse temperature.

To illustrate the SSB with an example, let us consider the angular momentum model Eq.(8) again, which shows a spontaneous breaking of  $O(3)$  symmetry. In the subspace with  $j = 1/2$ , the ground state is two-fold degenerate without  $H'$ , i.e.,  $|1/2, \pm 1/2\rangle$ . For a generic thermal state  $\rho = Z^{-1} \exp[-\beta H(\lambda)]$ , DoSS is shown to be  $S(O(3), \rho) = (3 + \cosh^{-1} \beta \lambda)/4$  [16], whose multi-valued natures at  $(T = 0, \lambda = 0)$  is shown in Fig. 3b. Specifically, when  $\beta\epsilon \gg 1$  while  $\lambda \neq 0$  the DoSS is nearly at

a constant value  $3/4$ . Then, by tuning the coupling  $\lambda$  to zero, the DoSS remains fixed at the same constant value (see (i) in Fig. 3b). In contrast, if one first fix the coupling  $\lambda = 0$  at the high temperature, the DoSS as a function of  $\beta$  from zero to infinity will follow the blue arrowed line (corresponding to the possess (ii)) in Fig. 3b. In the latter case, the DoSS at large  $\beta$  is unity. In analog to Eq.(13), in this example it is shown that

$$\lim_{\beta \rightarrow \infty} \lim_{\lambda \rightarrow 0} S(G, \rho) = 1, \quad \lim_{\lambda \rightarrow 0} \lim_{\beta \rightarrow \infty} S(G, \rho) = \frac{3}{4}. \quad (14)$$

Here,  $G = O(3)$ . On the other hand, the difference in DoSS reflected by above equations is also understood through inspecting on the final state, which is  $\rho_{f1} = |1/2, -1/2\rangle\langle 1/2, -1/2|$  or  $\rho_{f2} = 2^{-1} \sum_m |1/2, m\rangle\langle 1/2, m|$  upon limiting processes (i/ii). Clearly,  $\rho_{f1}$  is not invariant under the  $\pi$ -rotation that represented by the  $\sigma_x$  operation in the  $j = 1/2$  subspace, thus results in a DoSS smaller than unity.

*Conclusion.*— In this letter, we introduce a continuous measure, the degree of symmetry (DoS), for the symmetry of quantum system, which largely extrapolates the dichotomous approach of symmetry based on group representation theory. It is shown that the DoS possesses some good properties, such as basis-independent, invariant under the zero-point energy shifting as well as the scaling transformation. Since it can be expressed as an average of physical operators under the basis of irreducible representations for transformation groups, this measure is thus computable and detectable based on some quantum measurements.

In contrast to the previous explorations based on the abstract concepts of fuzz set [21] and transform information [22], our introduced DoS can feasibly open many applications in physics. As illustrated in this letter, the DoS is capable of identifying symmetry relevant phenomena and effects, such as the accidental level crossings and the spontaneous symmetry breaking. This, therefore, implies that the DoS could be a useful measure in related future studies, e.g., in characterizing systems near quantum criticality since it is closely related to the multi-level crossings [23].

We thank X. F. Liu, P. Zhang, X. G. Wang, S. X.

Yu, and L. P. Yang for helpful discussions. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11421063, 11534002, and 11475254) and the National 973 program (Grant Nos. 2014CB921403, 2012CB922104, and 2014CB921202).

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